Of Art and Math
Self-Similarity

You have wakened not out of sleep, but into a prior dream, and that dream lies within another, and so on, to infinity, which is the number of grains of sand. The path that you are to take is endless, and you will die before you have truly awakened
— **Jorge Luis Borges**

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The Argentinian author, Jorge Luis Borges, often wrote about his fear of *infinity*—the idea that space and/or time could continue forever. Though Borges' response may appear somewhat overblown, who amongst us has not felt a frisson of excitement when thinking of the infinite and our relative insignificance in front of it. Borges' quote of reality being a dream within a dream within a dream ad infinitum reminds us of the hall of mirrors effect—the seemingly infinite reflections one generates when one places two mirrors in front of each other—the same object over and over and over again.

This idea of infinite reflections can be seen in the chain ambigram in Figure 1 for the word *reflect.* In this design the "RE" and "FLECT" are written in a mirror-symmetric manner, which means that if we repeat this design over and over again it will read the same when held up against a mirror. (For a different ambigram for reflect, see *Introducing Symmetry,* in the March 2014 issue of *At Right Angles).*

Figure 1. An ambigram of "reflect," reflecting the infinite reflections in a pair of mirrors

This idea of repeating a similar shape (often at a different scale) over and over again, is called *self-similarity*. In other words, a self-similar image contains copies of itself at smaller scales. A simple example appears in Figure 2: a repeated pattern for a square that is copied, rotated and shrunk by a factor of $1/\sqrt{2}$.
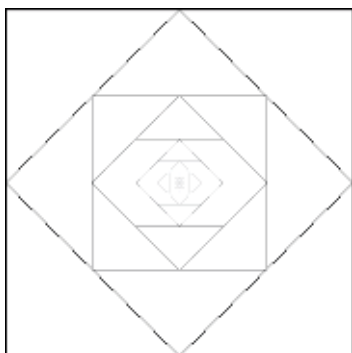

Figure 2. A self–similar design

Of course you can do this with typographical designs as well, such as the design for the word "Zoom" in Figure 3.
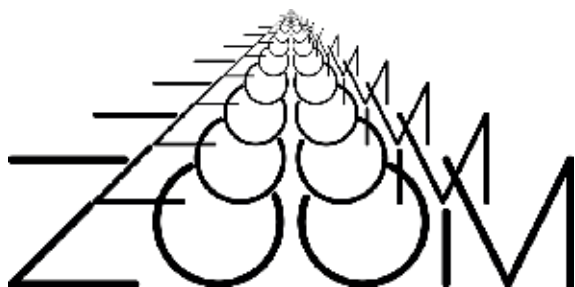

Figure 3. A self-similar ambigram for ZOOM

Examining self-similarity leads to a discussion of infinity, iteration and recursion, some of the ideas we discuss in this article.

Before taking a serious look at self-similarity, we present (see Figure 4) a rotational-ambigram of "self-similarity," which is *not* self-similar. However, below it is another version of the same design, where the word "self" is made up of little rotationally symmetric pieces of "self" and similarity is made up of little ambigrams of "similarity" and, most importantly the hyphen between the words is the complete ambigram for "self-similarity." So this leads to the question: What do you think the hyphen in the hyphen is made of?

Self similarity and Fractals

Self-similar shapes are all around us, from clouds to roots, from branches on trees to coastlines, from river deltas to mountains. The idea of self-similarity was popularized by Benoit B. Mandelbrot, whose 1982 book "The Fractal Geometry of Nature" showed how self-similar objects known as 'fractals' can be used to model 'rough' surfaces such as mountains and coastlines. Mandelbrot used examples such as these to explain how when you measure a coastline the length of the line would increase as you reduced the unit of measurement. Such convoluted folds upon folds that lead to increased length (or in the case of 3-d objects, increased surface area) can be seen in the structure of the alveoli in the human lungs as well as in the inside of our intestines. The volume does not increase by much, while the surface area increases without limit.

Figure 5 is an ambigram of "Fractal" which illustrates Mandelbrot's own definition of fractals: *A fractal denotes a geometric shape that breaks into parts, each a small scale model of the original*.
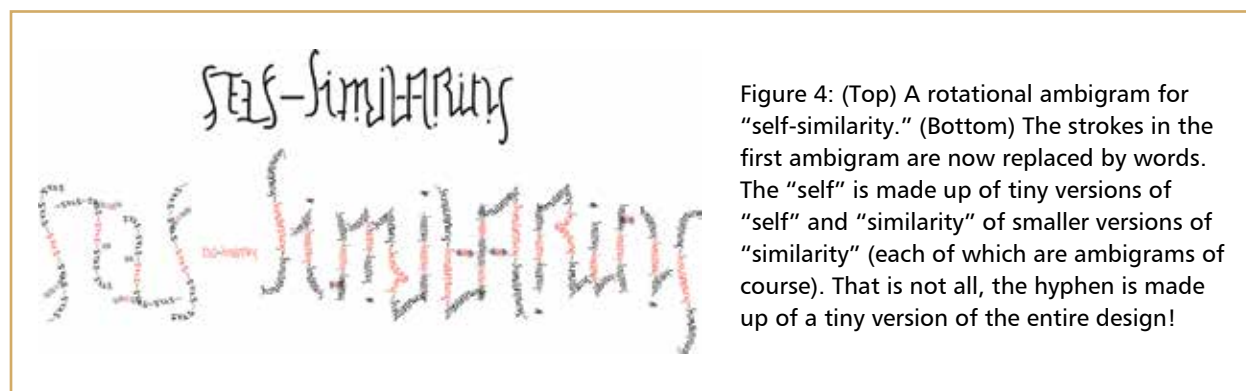

Figure 4: (Top) A rotational ambigram for "self-similarity." (Bottom) The strokes in the first ambigram are now replaced by words. The "self" is made up of tiny versions of "self" and "similarity" of smaller versions of "similarity" (each of which are ambigrams of course). That is not all, the hyphen is made up of a tiny version of the entire design!
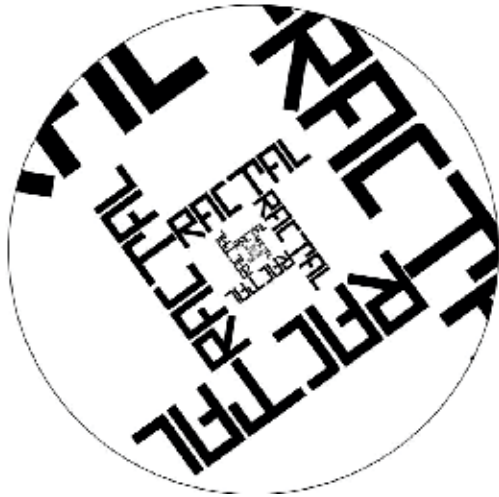
Figure 5: A self-similar, fractal ambigram for "Fractal"

In other words, fractals are geometrical shapes that exhibit invariance under scaling i.e. a piece of the whole, if enlarged, has the same geometrical features as the entire object itself. The design of Figure 6 is an artistic rendition of a fractal-like structure for the word "Mandelbrot".

Figure 6. A fractal ambigram for "Mandelbrot"

Speaking of Mandelbrot, what *does* the middle initial "B" in Benoit B. Mandelbrot stand for? A clue is provided in Figure 7.

Figure 7. Puzzle: What does the B in "Benoit B Mandelbrot" stand for?
Answer at the end of the article.

It is clear that the idea of infinity and infinite processes are an important aspect of fractals and self-similarity. We now examine the concept of infinity typographically and mathematically.

Infinity

Infinity means *without end*, or limitless. Mathematically speaking, a finite set has a definite number of elements. An infinite set is a set that is not finite. The word infinity is also used for describing a quantity that grows bigger and bigger, without limit, or a process which does not stop.

Figure 8 has two designs for "infinity" subtly different from each other. Notice how in the first design the chain is created by "in" mapping to itself and "finity" mapping to itself. In contrast the second design breaks the word up differently, mapping "ity" to "in" and "fin" to itself.

Figure 8. Two ambigrams for "infinity".
The first wraps around a circle and the second says infinity by word and symbol!

The first design wraps "infinity" around a circle. You can go round and round in a circle, and keep going on, so a circle can be said to represent an infinite path but in a finite and understandable manner. The second design is shaped like the symbol for infinity!

In keeping with the idea of self-similarity here are two other designs of the word "infinite". In fact there is a deeper play on the word as it emphasizes the *finite* that is *in* the *infinite.* The two designs in Figure 9 capture slightly different aspects of the design. The first focuses on mapping the design onto a sphere while the second is a self-similar shape that can be interpreted in two different ways. Either being made of an infinite repetition of the word "finite" *or* the infinite repetition of the word "infinite" (where the shape that reads as the last "e" in the word "finite" can be read as "in" in the word "infinite" when rotated by 90 degrees).

Infinities are difficult to grasp and when we try to apply the rules that worked with finite quantities things often go wrong. For instance, in an infinite set, *a part of the set can be equal to the whole*! The simplest example is the set of natural numbers, and its subset, the set of even numbers.

The set $\mathbb{N}$ = {1, 2, 3, 4, ...} of natural numbers is infinite. Now consider the set of even numbers $\mathbb{E}$ = {2, 4, 6, 8, ...} Clearly, the set of even numbers has half the number of elements of the set of natural numbers, doesn't it?

But not so quick! Things are tricky when it comes to infinite sets. We need to understand what it means for two sets to have an equal number of elements. Two sets have an equal number of elements when they can be put in one to one correspondence with each other. Think of children sitting on chairs. If each child can find a chair to sit on, and no chair is left over, then we know that each child *corresponds to* a chair, and the number of children is the same as the number of chairs.

Returning to the natural numbers, each number n in $\mathbb{N}$ corresponds to the number $2n$ in $\mathbb{E}$. So every element of $\mathbb{N}$ corresponds to an element of $\mathbb{E}$ and vice versa.

Thus though one set may intuitively look like it is half the other it is in fact not so! Our intuition is wrong, the sets $\mathbb{E}$ and N have the same number of elements. Since $\mathbb{E}$ is a part of $\mathbb{N}$, you can see that when it comes to infinite sets, *a part can be equal*

Figure 9. Two ambigrams for "infinite", a play on the finite in infinite. Is the second design an infinite repetition of the word "finite" or "infinite?"

to the whole. In fact, this part-whole equivalence has sometimes been used to define an infinite set.

Another interesting example where a part is equal to the whole, is provided by a fractal known as the Sierpinski Carpet.

The Sierpinski Carpet

The Sierpinski Carpet, like all fractals, is generated using the process of iteration. We begin with a simple rule and apply it over and over again.

Begin with a unit square, and divide the square into 9 equal parts. Remove the middle square. Now for each of the remaining 8 squares, we do the same thing. Break it into 9 equal parts and remove the middle square. Keep going on in this way till you get this infinitely filigreed Swiss-cheese effect. See Figure 10 for the first couple of steps and then the fifth stage of the carpet.
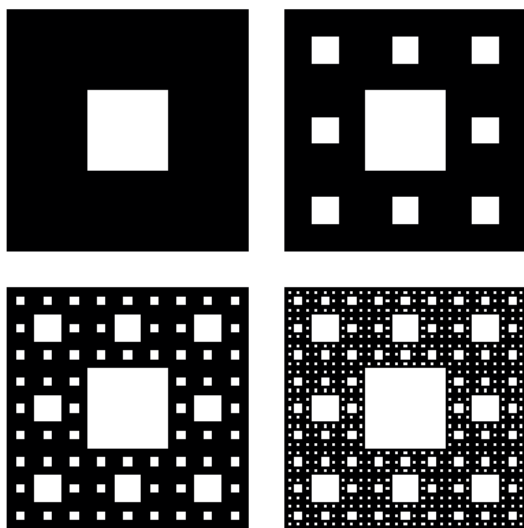

Figure 10. The Sierpinski Carpet

Which leads to the question: What is the total area of all the holes? Here is one way of computing the area of the holes in the Sierpinski Carpet. The first hole has area 1/9. In Step 2, you will remove 8 holes, each with area 1/9 th of the smaller square; so you will remove 8 holes with area $1/9^2$ or $8 \times \frac{1}{9^2} = 8/9^2$. In Step 3, for each of the smaller 8 holes, we remove 8 further holes with area $1/9^3$, so the area removed is $8^2/9^3$. In this manner it is easy to see that the total area of the hole is:

$$\frac{1}{9} + \frac{8}{9^2} + \frac{8^2}{9^3} + \frac{8^3}{9^4} + \cdots = \frac{1}{9}\left(1 + \frac{8}{9} + \frac{8^2}{9^2} + \frac{8^3}{9^3} + \cdots\right)$$

$$= \frac{1}{9} \times \frac{1}{1 - {}^8\!/_9} = 1.$$

To see why, we use the formula for the sum of the infinite Geometric Series:

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}, \text{ for } -1 < x < 1.$$

How crazy is that! The area of the holes (taking away just 1/9th of a square at a time) is equal to the area of the unit square! *Thus the hole is equal to the whole!*

This seemingly contradictory statement has inspired the following design—where the words *whole* and *hole* are mapped onto a square – with the letter o representing the *hole* in the Sierpinski carpet. Of course as you zoom in, the *whole* and *hole* keep interchanging. We call this design *(w)hole in One* (in keeping with the idea the area of the *hole* is equal to the *whole* of the unit square).
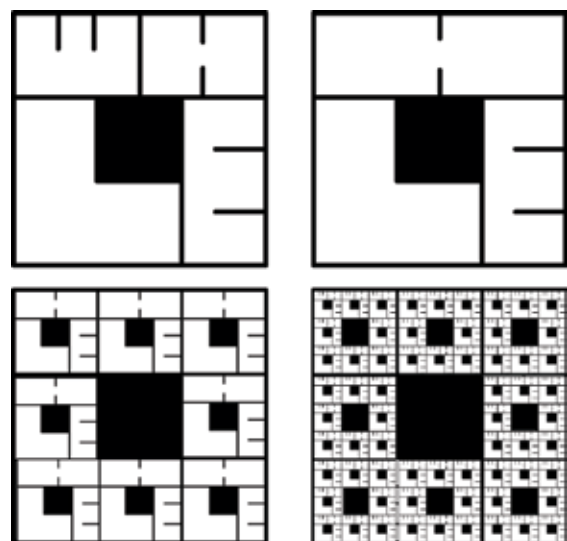

Figure 11. Fractal Ambigrams for "WHOLE" and "HOLE", a (w)hole in One.

The repetitive process of applying a set of simple rules that leads amazing designs like the Sierpinski carpet (and other fractal shapes) is called iteration.

Graphical interpretations of iteration

The process of iteration can be used to generate self-similar shapes. Graphically, we simply superimpose the original shape with a suitably scaled down version of the initial shape, and then repeat the process. The nested squares of Figure 2 is perhaps the simplest example of creating a self-similar structure using this process.

Essentially, such figures emerge from the repeated application of a series of simple steps—a program as it were, applied iteratively to the result of the previously applied rule. In this manner we can arrive at shapes and objects that are visually rich and complex.

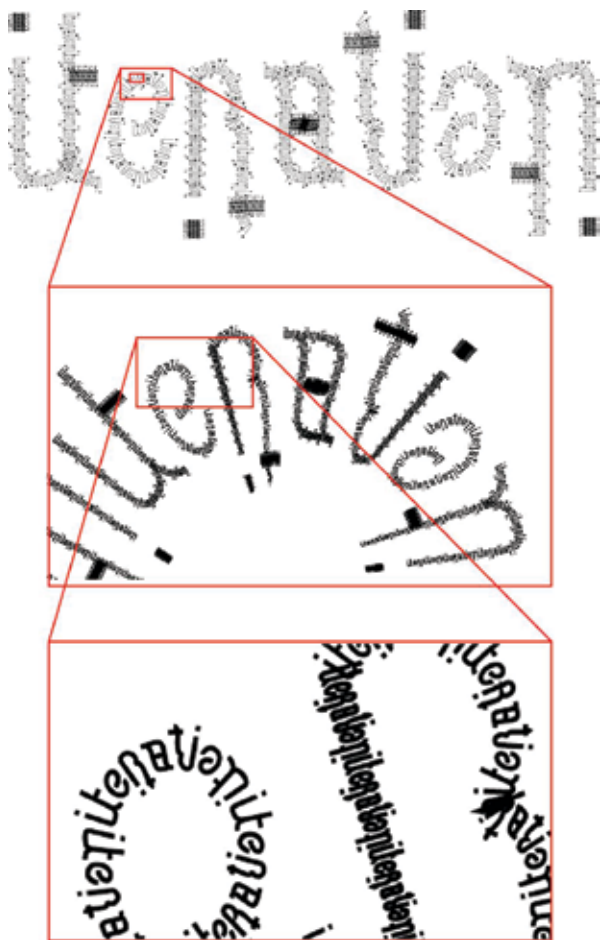Here is another, more creative, way to graphically interpret the idea of iteration.

Figure 12: An ambigram of "iteration", illustrating a graphical approach to a part can be equal to the whole.

At one level the first ambigram in Figure 12 can be read as a rotational ambigram for the word "iteration." However if you zoom into the design (see zoomed figures below) you will see that each of the strokes is made of smaller strokes that in turn spell *iteration*.

In fact you can go down one more level and see "iteration" all over again. Theoretically we could do this forever, (within the limits of computational technology and visual resolution of screen, print and eye!). A similar idea is explored in the design of the word "self-similarity" (Figure 4) specifically in the design of the hyphen.

There are other fascinating examples of such iterative techniques, one of which we examine next.

The Golden Mean

Another example of a mathematically and visually interesting structure is the Golden Rectangle (and its close relative the Golden Mean). The Golden Mean appears as the ratio of the sides of a Golden Rectangle. A Golden Rectangle is such that if you take out the largest square from it, the sides of the resulting rectangle are in the same ratio as the original rectangle. Suppose the sides of the Golden Rectangle are a and b, where b is smaller than a. The ratio a/b turns out to be the Golden Mean (denoted by ϕ). The largest square will be of side b. Once you remove it, the sides of the resulting rectangle are b and $a-b$. From this, it is easy to calculate the ratio a/b and find that it equals .

$$\phi = \frac{1+\sqrt{5}}{2} = 1.618033988 ....$$

If you begin with a Golden Rectangle and keep removing the squares, you will get a nested series of Golden Rectangles (see the underlying rectangles in Figure 14). The resulting figure shows self-similarity.

You may connect the diagonals using a spiral to obtain an approximation to what is called the Golden Spiral. Figure 13 shows an ambigram of "Golden Mean", placed in the form of a Golden Spiral inside a series of nested Golden Rectangles.
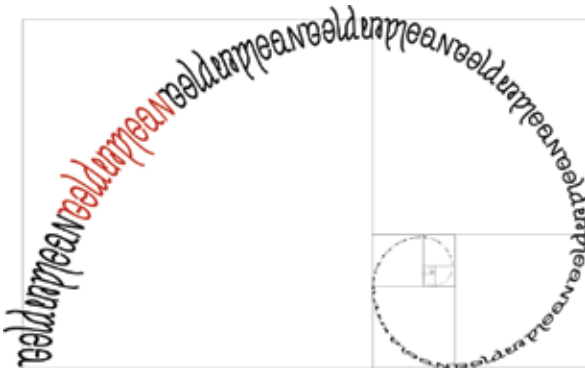
Figure 13. A rotationally symmetric chain-ambigram for the phrase "Golden Mean" mapped onto a Golden Spiral.

The Golden Mean appears in different contexts, in mathematics, in artistic circles, and even in the real world. It is closely related to the Fibonacci Numbers, namely 1, 1, 2, 3, 5, 8, 13, 21, ... Note that the Fibonacci Numbers begin with 1 and 1, and then each number the sum of the previous two numbers. If you take the ratio of successive Fibonacci numbers, the ratio converges to the Golden Mean.

The Fibonacci numbers are an example of a recursively defined sequence, where a few initial terms are defined, and then the sequence is built up by using the definition of the previous term (or terms).

Recursion and Pascal's Triangle

Recursion is similar to iteration. While iteration involves applying a simple rule to an object repeatedly, like in the creation of the Sierpinski Carpet, recursion involves using the results of a previous calculation in finding the next value, as in the definition of the Fibonacci numbers.

Fractals are usually obtained by iteration. Thus it is rather surprising that the fractal of Figure 14, called the Sierpinski triangle, may also be obtained using a recursive process.

The triangle in Figure 14 is a binary Pascal's triangle, where you use binary arithmetic (where $0 + 0 = 0$; $0 + 1 = 1$ and, $1 + 1 = 0$) to create the Pascal's triangle. The recursion is as follows: Each row and column begins and ends with a 1. Every other number is found by the (binary) addition of numbers above it. The formula for the recursion is

$$F(n + 1, k) = F(n, k - 1) + F(n, k)$$

where $F(n,k)$ is the term in the nth row and kth column, for $n = 0, 1, 2, 3, ...$ and $k = 0, 1, 2, 3, ...$ and the rules of binary arithmetic are used. In addition, we need the following values:

$$F(n, 0) = 1 = F(n,n).$$

This recurrence relation is the recurrence for generating Pascal's Triangle, satisfied by the Binomial coefficients.
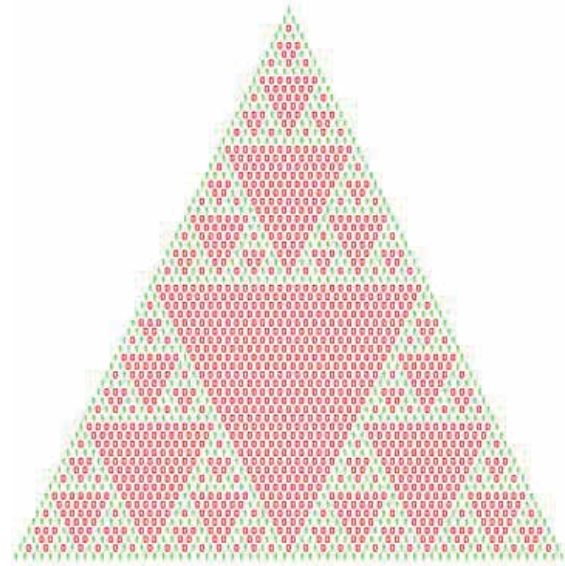

Figure 14. The binary Pascal's Triangle is also the Sierpinski Triangle

Of course, you can guess how to obtain the Sierpinski Triangle by iteration. Begin with a triangle, remove the middle triangle in step 1, which will leave behind three triangles to which you do the same! And just repeat this process forever.

The fact that Pascal's triangle is symmetric upon reflection, led to the design below (Figure 15)—made up of row over row of mirror-symmetric designs for the word "Pascal" increasing in size as we go down the rows. We call this design "*a* Pascals Triangle" (a triangle *made up of many* "Pascals") as opposed to "*the* Pascal's triangle" (the triangle *of or belonging to* Pascal). (Author's note: This design was created under psignificant work pressure. Can you guess why?)

Figure 15. An ambigrammist's approach to Pascals Triangle (as opposed to Pascal's Triangle). What a difference an apostrophe makes!

Figure 16. An ambigram for "Hidden Beauty", whose beauty is not hidden from anyone!

In conclusion

We have explored many ideas in this article—self-similarity, iteration, recursion, infinity, and attempted to represent them graphically even while connecting them to deeper mathematical ideas. We hope that like us, you too experienced many feelings when you encountered these ideas—feelings of wonder, amusement, surprise,

or the indescribable feeling when one finds something beautiful. We hope that these feelings make you wish to create something new, and perhaps dream up interesting things to share with your friends. As Borges eloquently said, "The mind was dreaming. The world was its dream." There is a lot of beauty one can find, hidden away in the world of ideas. We close with an ambigram for "Hidden Beauty" in Figure 16, where the word *hidden* becomes *beauty* when rotated 180 degrees!

Answer to the Puzzle in Figure 7: The "B" in "Benoit B Mandelbrot" stands for Benoit B Mandelbrot... and so on forever! Here is another way of representing the same idea, that we call, "Just let me B."

Figure 17. "Just let me B:" A fractal design for Benoit B Mandelbrot designed to answer the question "what does the B in Benoit B Mandelbrot stand for?"

PUNYA MISHRA, when not creating ambigrams, is professor of educational technology at Michigan State University. GAURAV BHATNAGAR, when not teaching or doing mathematics, is Senior Vice-President at Educomp Solutions Ltd. They have known each other since they were students in high-school.

Over the years, they have shared their love of art, mathematics, bad jokes, puns, nonsense verse and other forms of deep-play with all and sundry. Their talents however, have never truly been appreciated by their family and friends.

Each of the ambigrams presented in this article is an original design created by Punya with mathematical input from Gaurav. Please contact Punya if you want to use any of these designs in your own work.

To you, dear reader, we have a simple request. Do share your thoughts, comments, math poems, or any bad jokes you have made with the authors. Punya can be reached at punya@msu.edu or through his website at http://punyamishra.com and Gaurav can be reached at bhatnagarg@gmail.com and his website at http://gbhatnagar.com/.