Of Art and Mathematics

Paradoxes:

Part 2 of 2

feature

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This is not the first sentence of this article.

The above sentence can be both true and false. It is clearly the first sentence of *this* article. So it is false, because it says it is not the first sentence! But because this is part 2 of our article on Paradoxes, if we regard both parts as one article, it is true! We leave it to you to resolve this paradox.

In the first part of this two-part exposition on paradoxes in mathematics, we introduced the idea of self-reference, the nature of mathematical truth, the problems with circular proofs and explored Zeno's Paradox. In this part we delve deeper into the challenges of determining the 'truth value' of pathological self-referential statements, visual paradoxes and more.

Self - Reference and Russell's Paradox

There is a class of paradoxes that arise from objects referring to themselves. The classic example is Epimenides Paradox (also called the Liar Paradox). Epimenides was a Cretan, who famously remarked "All Cretans are liars." So did Epimenides tell the truth? If he did, then he must be a liar, since he is a Cretan, and so he must be lying! If he was lying, then again it is not the case that all Cretans are liars, and so

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Figure 1. An ambiguous design that can be read as both “true” and “false.”

he must be telling the truth, and that cannot be! Figure 1 is an ambiguous design that can be read as both “true” and “false.”

The artwork of M.C. Escher (such as his famous illustration that shows two hands painting each other) provides many visual examples of such phenomena. Another older analogy or picture is that of the *ouroboros*—an image of a snake eating its own tail (how’s *that* for a vicious circle!). An ambigram of *ouroboros* was featured in our first article on paradoxes.

Here is another variation of the Liar Paradox. Consider the following two sentences that differ by just one word.

This sentence is true.

This sentence is false.

The first is somewhat inconsequential – apart from the apparent novelty of a sentence speaking to its own truth value.

The second, however, is pathological. The truth and falsity of such pathologically self-referential statements is hard to pin down. Trying to assign a truth value to it leads to a contradiction, just like in the Liar Paradox. Figure 2 is a rotational ambigram that reads “true” one way and “false” when rotated 180 degrees.

Figure 2: Rotational ambigram that reads “False” one way and “True” the other. (This design was inspired by a design by John Longdon.)

A variant of this (that does not employ self-reference) is also known as the Card paradox or Jourdain’s paradox (named after the person who developed it). In this version, there is a card with statements printed on both sides. The front says, “The statement on the other side of this card is TRUE,” while the back says, “The statement on the other side of this card is FALSE.” Think through it, and you will find that trying to assign a truth value to either of them leads to a paradox!

Figure 3 combines the liar’s paradox and Jourdain’s paradox (in its new ambigram one-sided version) into one design.

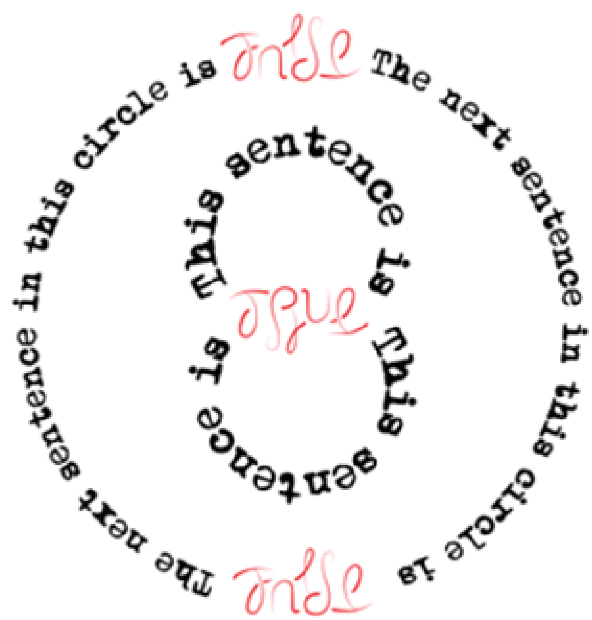

Figure 3: Two paradoxes in one. Inside the circle is the ambigram for the pair of sentences “This sentence is True/This sentence is False”. The outer circle is an original design for the Jourdain two-sided-card paradox, which can, due to the magic of ambigrams, be reduced to being printed on just one side!

Another interesting example is the sentence: "This sentence has two ewrrors." Does this indeed have two errors? Is the error in counting errors itself an error? If that is the case, then does it have two errors or just one?

What is intriguing about the examples above is that they somehow arise because the sentences refer to themselves. The paradox was summarized in the mathematical context by Russell, and has come to be known as Russell's paradox. Russell's paradox concerns sets. Consider a set R of all sets that do not contain themselves. Then Russell asked, does this set R contain itself? If it does contain itself, then it is not a member of R. But if it is not a member of R, then it does contain itself.

Russell's Paradox was resolved by banning such sets from mathematics. Recall that one thinks of a set as a *well-defined* collection of objects. Here by well-defined we mean that given an element a and a set A, we should be able to determine whether a belongs to A or not. So Russell's paradox shows that a set of all sets that do not contain themselves is not well-defined. By creating a distinction between an element and a set, such situations do not arise. You could have sets whose members are other sets, but an element of a set cannot be the set itself. Thus, in some sense, self-reference is not allowed in Set Theory!

Visual contradictions

Next, we turn to graphic contradictions, where we use ambigrams to create paradoxical representations.

Figure 4: A somewhat inelegant design that captures a visual paradox – the word "asymmetry" written in a symmetric manner.

Figure 4 shows an ambigram for asymmetry, but it is symmetric. So in some sense, this design is a *visual contradiction*! But it is not a very elegant solution – which in some strange way is appropriate.

Recall the idea of self-similarity from our earlier column, where a part of a figure is similar to (or a scaled-down version of) the original. Here is an ambigram for similarity which is made up of small pieces of self (Figure 5). Should we consider this to be self-similarity?

Another set of visual paradoxes have to do with the problems that arise when one attempts to represent a world of 3 dimensions in 2 dimensions – such as in a painting or drawing. The Dutch artist M.C. Escher was the master at this. His amazing paintings often explore the paradoxes and impossible figures that can be created through painting. For instance, he took the mathematician and physicist Roger Penrose's image of an impossible triangle and based some of his work on it (Figure 6).

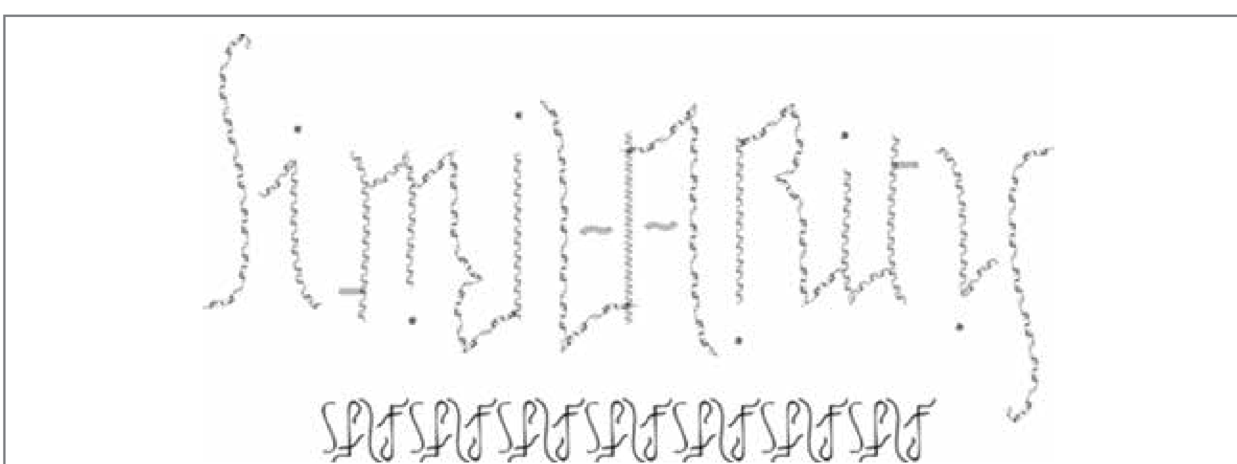

Figure 5: Here is an ambigram for Similarity which is made up of small pieces of Self. Should we consider this to be self-similarity?

Figure 6. A Penrose Triangle – a visual representation of an object that cannot exist in the real world.

As homage to M.C. Escher, we present below (Figure 7) a rotational ambigram of his name written using an impossible font!

Figure 7: Rotationally symmetric ambigram for M.C. Escher written using an impossible alphabet style.

As it turns out, the Penrose Triangle is also connected to another famous geometrical shape, the Möbius strip. A Möbius strip has many interesting properties, one of which is that it has only one side and one edge (Figure 8).

Puzzle: What is the relationship between a Penrose Triangle and a Möbius strip?

Figure 8. An unending reading of the word Möbius irrespective of how you are holding the paper!

Another famous impossible object is the "impossible cube." The impossible cube builds on the manner in which simple line drawings of 3D shapes can be quite ambiguous. For instance, see the wire-frame cube below (also known as the Necker Cube). This image usually oscillates between two different orientations. For instance, in Figure 9, is the person shown sitting *on* the cube or magically stuck to the ceiling *inside* it?

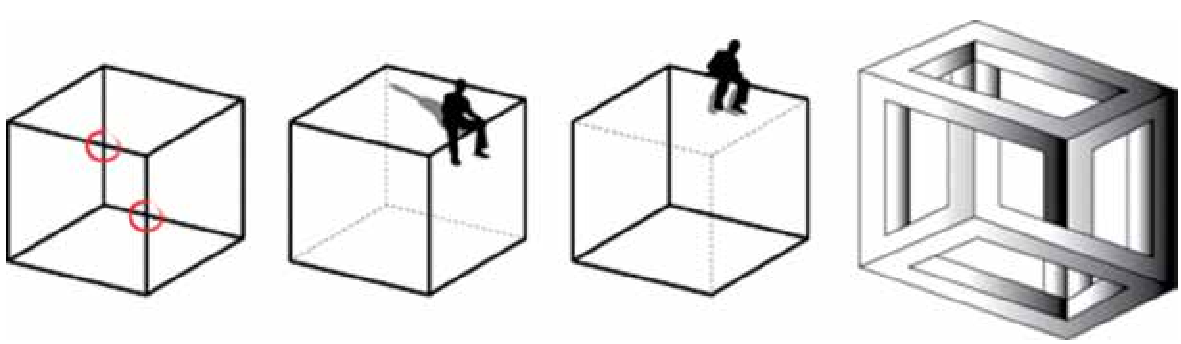

Figure 9. The Necker Cube – and how it can lead to two different 3D interpretations and through that to an impossible or paradoxical object.

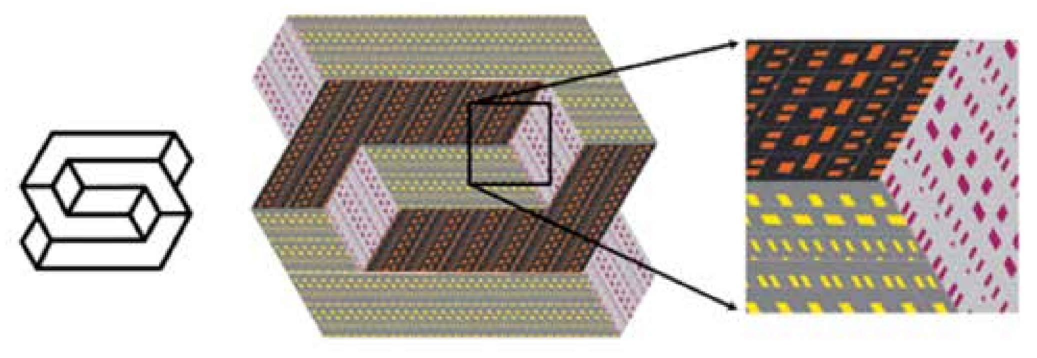

Figure 10: The impossible cube? In this design the word “cube” is used to create a series of shapes that oscillate between one reading and the other.

Figure 11: An impossible typeface based on the Necker Cube and Penrose Triangle. Spelling the word “Illusions.”

These images oscillate between two opposite incommensurable interpretations, somewhat like the liar paradoxes we had described earlier. Figure 10 is another ambiguous shape that can be read two ways! What is cool about that design is that each of these shapes is built from tiny squares that read the word “cube.”

These representations fool our minds to see things in ways that are strange or impossible. These are visual paradoxes, or illusions, as reflected in the design in Figure 11, which is the word “illusions” represented using an impossible font (akin to the Penrose Triangle or Necker Cube).

Mathematical Truth and the Real World

One of the most fundamental puzzles of the philosophy of mathematics has to do with the fact that though mathematical truths appear to have a compelling inevitability (from axiom to theorem via proof) and find great applicability in the world, there is little we know of why this is the case. The physicist Wigner called it the “unreasonable effectiveness of mathematics” to explain, understand and predict the phenomena in the real world. The question is how something that exists in some kind of an “ideal” world can connect to and make sense in the “real” world we live in.

Figure 12: The Ideal-Real ambigram, representing the paradoxical thought that the Real world often appears to be a reflection of the Ideal mathematical theory!

Figure 12 maps the word “ideal” to “real.” Is the ideal real – and real just a mere reflection of the ideal? Or vice versa?

Clearly this is not an issue that will be resolved anytime soon – but it is intriguing to think about.

So with that, we bid adieu, but before we depart we would like to bring you the following self-serving public announcement.

This is the last sentence of the article. No this is. This.

Answer to Puzzle:
The Möbius Strip and the Penrose Triangle have an interesting relationship to each other. If you trace a line around the Penrose Triangle, you will get a 3-loop Möbius strip. M.C. Escher used this property in some of his most famous etchings.

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Over the years, they have shared their love of art, mathematics, bad jokes, puns, nonsense verse and other forms of deep-play with all and sundry. Their talents, however, have never truly been appreciated by their family and friends.

Each of the ambigrams presented in this article is an original design created by Punya with mathematical input from Gaurav (except when mentioned otherwise). Please contact Punya if you want to use any of these designs in your own work.

To you, dear reader, we have a simple request. Do share your thoughts, comments, math poems, or any bad jokes you have made with the authors. Punya can be reached at punya@msu.edu or through his website at http://punyamishra.com and Gaurav can be reached at bhatnagarg@gmail.com and his website at http://gbhatnagar.com/.